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Sensitivity of an Annual Mean Diffusive Energy Balance Model With an Ice Sheet

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The sensitivity of a diffusive energy balance model that contains a simple ice sheet is compared with that of a model with snow cover only. The effect of the elevated ice sheet surface on the radiative cooling is calculated by using a radiative transfer model. Because the temperature in the atmosphere decreases with height, the main effect of the ice sheet elevation is to reduce the outgoing infrared radiation. This reduction in the radiative cooling decreases the sensitivity of the ice sheet size to changes in the solar constant by partially counteracting the albedo feedback. For a reasonable choice of parameters, this effect can reduce the strength of the albedo feedback by a factor of 2.

1. Introduction

Zonally averaged energy balance models have received much attention since they were introduced by *Budyko* [1969] and *Sellers* [1969]. One reason is that the extent of permanent snow cover in their models is very sensitive to changes in insolation. Budyko and Sellers both found that a reduction in the solar constant of only ~2% was sufficient to cause a completely ice covered earth. This large response for a small change in solar constant is caused by albedo feedback, which operates in the following way. If the solar constant is reduced slightly, the result is a small cooling and an increase in the amount of snow cover. The snow, with its higher albedo, reflects more sunlight, leading to further cooling and a further increase in snow cover, thus amplifying the effect of the reduction in solar constant. The strength of the amplification depends in part on the relative albedos of snow-free and snow-covered areas. The larger the difference in albedos, the greater the reduction in solar heating when the snow cover expands. If the albedos of snow-covered and snow-free areas were the same, there would be no albedo feedback. The albedo feedback has recently been discussed in a number of papers, most of which conclude that the effect is significant, but not as large as in the Budyko or Sellers models. For a review of energy balance models and a discussion of albedo feedback, see *North et al.* [1981].

Models have also been developed to study the relation between ice sheets and climate, beginning with the studies of ice sheet dynamics by *Weertman* [1961, 1962, 1976]. Other climate models that include the cryosphere have been proposed [*Källen et al.*, 1979; *Sergin*, 1979], including a diffusive energy balance model with a highly simplified continental ice sheet [*Pollard et al.*, 1980] and a diffusive model with a power flow law ice sheet model [*Birchfield et al.*, 1982]. These models are improving understanding of the causes of the Quaternary ice ages. For this reason it is important to understand the effect of ice sheets on the model response before attempting to simulate the historical record of the ice ages.

The purpose of this paper is to discuss the sensitivity of an annual mean diffusive energy balance model which includes a large ice sheet. In the model proposed here, the presence of the ice sheet affects only the infrared cooling parameterization. The height of the ice surface lowers the effective radiating temperature of an atmospheric column. The resulting decrease in the radiative cooling to some extent counteracts the albedo feedback and reduces the sensitivity of the ice cover to changes in solar constant.

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2. Energy Balance Model

The energy balance equation for annual mean insolation forcing can be written

$$\frac{D}{\cos\theta}\frac{d}{d\theta}\cos\theta\frac{dT(\theta)}{d\theta} = -Qs(\theta)\alpha[T(\theta)] + E[T(\theta)] \qquad (1)$$

This model is the same as that of *North* [1975]. The physical processes incorporated in the model are shown schematically in Figure 1. The dependent variable T is the zonal mean sea level temperature, which is a function of latitude θ.

The left-hand side of (1) is the convergence of the meridional heat flux. The heat flux is parameterized as the product of a constant thermal diffusivity D and the meridional temperature gradient $dT/d\theta$. This diffusive approach has been used in many energy balance models. The diffusive parameterization is used here for simplicity and for comparison with earlier models.

The boundary conditions on (1) are that the flux vanish at the equator and at the pole

$$\cos\theta\frac{dT}{d\theta} = 0 \qquad \text{at} \qquad \theta = 0, \frac{\pi}{2} \qquad (2)$$

The first term on the right-hand side in (1) is the solar heating. Q is the solar constant; the standard value Q_0 is taken here to be 1376 W m^{-2}. The latitudinal distribution of annual mean insolation is $s(\theta)$. This function is plotted in Figure 1 in *Held and Suarez* [1974]. The fraction of the solar radiation absorbed at a given latitude depends on the temperature. For a model with snow cover only, the step function form for the absorptivity suggested by *Budyko* [1969] is used:

$$\alpha[T(\theta)] = \begin{cases} \alpha_1 = 1 - \text{snow-free albedo} & T > T_0 \\ \alpha_2 = \alpha_1 - \alpha' = 1 - \text{snow albedo} & T \le T_0 \end{cases} \qquad (3)$$

The parameterization of the absorptivity for a model with an ice sheet will be discussed in section 3. The surface is snow covered poleward of the latitude θ_0 where the temperature is T_0. The difference in absorptivity between snow-covered

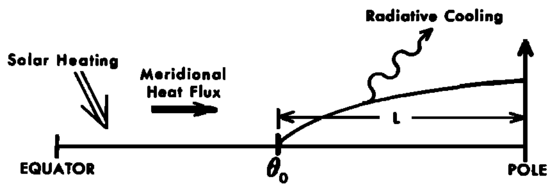

Fig. 1. Schematic of the physical processes included in the model. The details of the various processes are described in the text.

and snow-free surfaces is α'. The snow-free and the snow-covered albedos are set to 0.29 and 0.59, respectively, yielding $\alpha_1 = 0.71$, $\alpha_2 = 0.41$, and $\alpha' = 0.3$. These values are similar to those used in many energy balance models [*cf. Oerlemans and van den Dool,* 1978] and are used here mainly for comparison.

The last term in (1) is the radiative cooling E. A linear function of the form $E = a + bT$ has been used in many energy balance models [*North et al.,* 1981]. This general form has also been used in this study, with the inclusion of height dependence in the coefficients a and b. If this linear form is used for the radiative cooling, along with the absorptivity parameterization (3), (1) can be solved analytically for the temperature as a function of latitude. Solutions to (1), along with linear stability conditions, were found for diffusive models by *Held and Suarez* [1974], *North* [1975], and *Cahalan and North* [1979].

In this paper the sensitivity of the model will refer to the response of the snowline to changes in the solar constant, defined as $\partial\theta_0/\partial Q$. The sensitivity depends on the value of D/b and the difference in the absorptivity between snow-covered and snow-free surfaces. Smaller values of b result in a more sensitive model. The coefficients of the radiative cooling parameterization are therefore important in determining the overall sensitivity of the model.

To establish the infrared cooling parameterization, it is necessary to know the outgoing infrared irradiance at the top of the atmosphere, E. There are two methods for determining E. The first is to measure E in situ by using satellite-borne radiometers. The second is to calculate E directly by using a radiative transfer model. In either case, E is regressed against surface temperature to find the coefficients a and b. Different approaches to using the satellite data can be found in *Held and Suarez* [1974], *Cess* [1976], and *Oerlemans and van den Dool* [1978]. This type of parameterization has been used in most one-level energy balance models. If ice sheets are included in energy balance models, however, the height of the ice sheet, which may be several kilometers, may have a significant effect on the radiative cooling. It is difficult, however, to determine the explicit dependence of the radiative cooling on the surface elevation from the satellite data alone. Oerlemans and van den Dool found that using surface temperature was a better way to predict E than using sea-level temperature, but the number of points at high elevations was small. Using a radiative transfer model allows one to vary temperatures and surface elevation independently. The calculations described below were done with the radiative transfer model of *Stone and Manabe* [1968].

Two different approaches to calculating the radiative cooling coefficients with the radiative transfer model may be used. The first could be called the 'climatological' approach. *Budyko* [1969] used this approach when he took monthly mean temperature, humidity and cloudiness profiles from many different stations and calculated the radiative cooling by the atmosphere at each station. From these calculations he found that E was best fit when $a = 201.6$ W m^{-2} and $b = 1.45$ W m^{-2} °C^{-1}, assuming no variation in cloudiness. This value of b is quite low and leads to the sensitivity mentioned in section 1.

Similar calculations were done with the radiative transfer model for three climatological atmospheric temperature profiles: the annual mean, zonal mean temperatures at 45°, 60°, and 75°N, taken from *Oort and Rasmusson* [1971]. Cloudiness and relative humidity were the same for each profile, while the annual mean, zonal mean ozone concentration was taken from *Fels et al.* [1980] for each profile. The three temperature profiles are plotted in Figure 2. The outgoing IR irradiance E calculated for each profile is plotted as a function of surface temperature in Figure 3. The coefficients of the best-fit straight line to these points are $a = 215.6$ W m^{-2} and $b = 1.46$ W m^{-2} °C^{-1}. The coefficients calculated from these profiles are very close to those found by Budyko, especially b. Either of these radiative cooling parameterizations would make for a very sensitive model. By contrast, the satellite measurements used by *Oerlemans and van den Dool* [1978] are also plotted in Figure 3 as a function of surface temperature. The coefficients they derived for the radiative cooling parameterization are $a = 205$ W m^{-2} and $b = 2.23$ W m^{-2} °C^{-1}. This value of b is ~50% larger than that calculated by using the observed temperature profiles in the radiative transfer model, and would lead to a much less sensitive energy balance model. The differences between the radiative transfer model results and the satellite observations are presumably the result of the assumption that cloudiness was constant in the model.

The second approach to calculating the radiative cooling coefficients could be termed the 'perturbation' approach. In

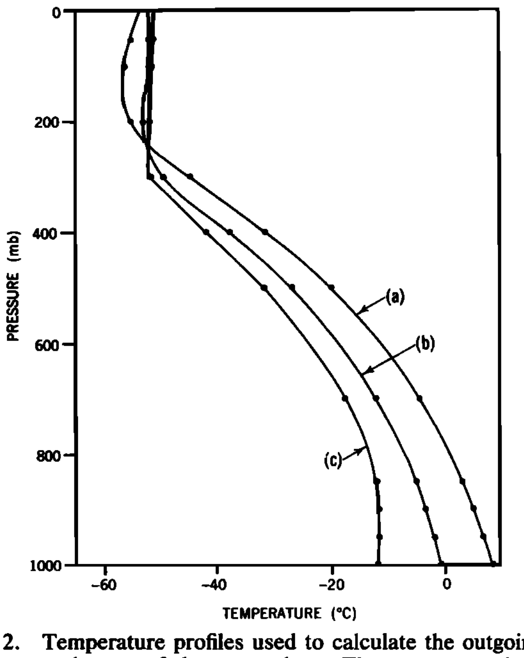

Fig. 2. Temperature profiles used to calculate the outgoing IR irradiance at the top of the atmosphere. These are annual-mean, zonal-mean temperatures taken from *Oort and Rasmusson* [1971]; (*a*) at 45°N, (*b*) at 60°N, and (*c*) at 75°N. Dots indicate observations. Temperatures between observations were calculated by using cubic interpolation.

this approach, the climatological temperature profiles are perturbed slightly and the changes in the radiative cooling calculated with the radiative transfer model. The results depend on the choice of the vertical structure of the temperature perturbation. Two different perturbations have been used in these calculations: a temperature change that decreases linearly with pressure to zero at 200 mbar, and a temperature change constant with height. The general circulation model calculations of *Wetherald and Manabe* [1975] suggest that the first of these is more appropriate in high latitudes (see Figure 3 in Wetherald and Manabe). Therefore, the first perturbation has been used for the calculations described in the text. In middle latitudes the temperature change is more nearly constant with height, so results for the second form of the perturbation are included in the appendix.

The perturbed temperature profiles are shown in Figure 4. As can be seen by comparing Figure 4 with Figure 2, the different profiles produced by using this form of perturbation resemble the different climatological profiles. By using the climatological profile from 45°N as a basis, E is calculated for each perturbed profile and plotted as a function of surface temperature in Figure 5. The resulting regression coefficients are a = 215.8 W m^{-2} and b = 1.52 W m^{-2} °C^{-1}. These coefficients are very close to those computed by using Budyko's 'climatological' approach. Thus, this form of perturbation closely approximates the differences between climatological temperature profiles from different latitudes. The coefficients calculated by using the perturbed climatological profiles from 60° and 75°N are very close to those for the profile from 45°N. Therefore, the radiative cooling parameterization has no explicit dependence on latitude. The perturbation method has been used instead of the 'climatological' method when examining the effects of surface elevation changes.

Changes in the surface elevation are easily simulated in the radiative transfer model by changes in the surface

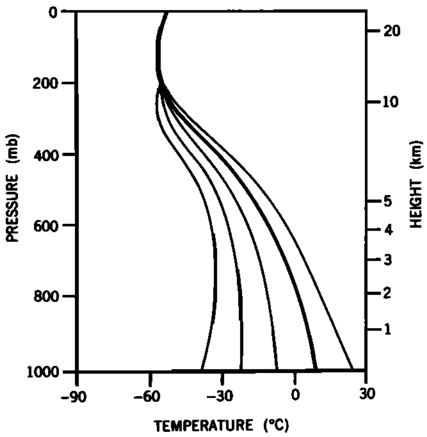

Fig. 4. Climatological temperature profile from 45°N (heavy line) and perturbed profiles (light lines).

pressure p_*. Raising the surface (lowering the surface pressure) simply eliminates that part of the atmosphere below the new surface pressure. When changing the surface elevation, assumptions must be made about the response of cloud heights and amounts. Therefore, the clouds are handled in two alternative ways to try to gauge the importance of a particular assumption about clouds. In the first method, the cloud amount of the ith cloud is initialized at pressure p_i. When the surface pressure is changed, clouds remain at constant σ levels ($\sigma_i = p_i/p_*$). In this case, the clouds move to lower pressures as the surface is raised, and cloud amounts do not change. In the second method, clouds remain fixed at constant pressure levels. Cloud amounts are initialized when the surface elevation is zero. When the surface is raised, the surface pressure is decreased, and all clouds that are at higher pressures than the new surface pressure are removed. The surface gradually eclipses succeeding cloud layers as the surface elevation increases.

For each surface elevation, the temperature profile can be perturbed as described above and the coefficients a and b calculated. The relationship between E calculated from the model and the surface temperature T_* is very linear in every case. The coefficients for different surface elevations and the

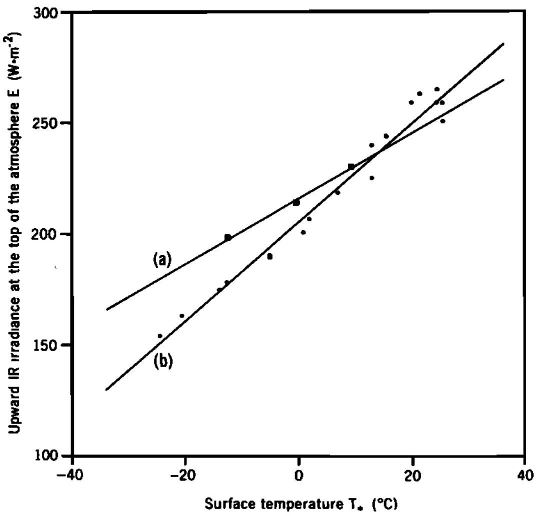

Fig. 3. Outgoing IR irradiance at the top of the atmosphere E as a function of surface temperature T_*. Squares indicate values of E calculated for each of the temperature profiles in Figure 2. Line a is the least squares fit to those three points. Dots indicate satellite measurements of zonal mean outgoing IR irradiance at the top of the atmosphere for 10° latitudinal hands from *Ellis and Vonder Haar* [1976], and line b is the fit derived by *Oerlemans and van den Dool* [1978].

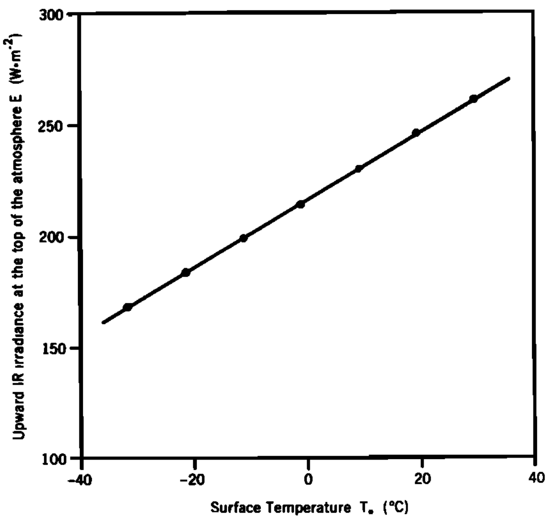

Fig. 5. Outgoing IR irradiance E as a function of surface temperature T_* calculated from the perturbed temperature profiles in Figure 4.

TABLE 1. Regression Coefficients for a Linear Fit to the Radiative Cooling E as a Function of Surface Temperature T_*

Surface Elevation h, km	a, W m^{-2}	b, W m^{-2} °C^{-1}
	Clouds at Constant σ Levels	
0	215.8	1.52
1	214.8	1.56
2	213.6	1.59
3	212.2	1.61
4	210.6	1.60
5	208.7	1.57
	Clouds at Constant Pressure Levels	
0	215.8	1.52
1	218.1	1.64
2	224.1	1.86
3	226.7	1.95
4	226.8	1.93
5	228.1	1.94

The climatological temperature profile is perturbed as shown in Figure 4.

two different cloud treatments are listed in Table 1. Depending on the method used, the coefficients may or may not depend on height. For the case with clouds at constant σ levels both a and b change little with elevation. In the other case, with clouds at constant pressure levels, a and b show larger changes as the surface elevation increases. For this case there is less cloud cover when the surface is higher and, as a result, the clouds absorb less of the upward radiation from the surface and radiate less from their tops. Therefore, the surface radiates more freely, and the total radiation is larger. The clouds also contribute a smaller fraction to the total upward radiation so the upward radiation is more dependent on the surface temperature, hence b is larger. The height dependence of the coefficients can be most simply represented by equations of the form $a = a_0 + a_1 h$ and $b = b_0 + b_1 h$, where h is the surface elevation. For the first case with clouds at constant σ levels $a_0 = 216.2$ W m^{-2}, $a_1 = -1.41$ W m^{-2} km^{-1}, $b_0 = 1.55$ W m^{-2} °C^{-1}, and $b_1 = 0.01$ W m^{-2} °C^{-1} km^{-1}. For the case with clouds at constant pressure levels $a_0 = 216.8$ W m^{-2}, $a_1 = 2.58$ W m^{-2} km^{-1}, $b_0 = 1.59$ W m^{-2} °C^{-1}, and $b_1 = 0.09$ W m^{-2} °C^{-1} km^{-1}.

A general form for the radiative cooling is

$$E(T_*, h) = (a_0 + a_1 h) + (b_0 + b_1 h)T_* \tag{4}$$

The surface temperature T_* can be calculated from the sea-level temperature T in the energy balance model by assuming a constant lapse rate Γ. Then $T_* = T - \Gamma h$ and

$$E(T, h) = (a_0 + a_1 h) + (b_0 + b_1 h)(T - \Gamma h) \tag{5}$$

The net effect of the height can be seen by taking the partial derivative of E with respect to h

$$\frac{\partial E}{\partial h} = a_1 - b_0\Gamma + b_1 T - 2b_1\Gamma h \tag{6}$$

Using either of the two assumptions about clouds, the largest term is $b_0\Gamma$. This term is present even if the radiative cooling coefficients do not depend on height (i.e., a_1 and $b_1 = 0$). The effect of this term is to decrease the radiation as the surface elevation increases because the surface becomes colder. The elevation of an ice sheet is important to the radiative cooling because the surface temperature should be used to calculate the radiative cooling, and the surface temperature is colder than the sea-level temperature. The other terms in (6) are the results of changes in the radiating properties of the atmosphere with height. These terms are small and generally cancel to some degree.

As was pointed out by one of the reviewers, one can test the existence of the effect by examining the satellite radiation statistics over the Greenland and Antarctic ice sheets. The outgoing IR is mapped in Figure 1 of *Hartmann and Short* [1980]. In both cases, the outgoing IR decreases inland from the edge of the ice sheet as the ice thickness increases. Over Greenland the IR irradiance is ~20 to 40 W m^{-2} less than at lower elevations at the same latitude. An estimate of the decrease predicted by the model would be $b_0\Gamma h \cong$ (1.55 W m^{-2} °C^{-1}) (6.5 °C km^{-1}) (3 km) $\cong$ 30 W m^{-2}. The minimum in the outgoing IR over Antarctica can be seen in Figure 16 of *Raschke et al.* [1973] to lie over the center of the East Antarctic ice sheet, not over the pole, which is at a lower elevation (2800 m versus 4000 m). The satellite measurements seem to support the model calculations.

3. Ice Sheet Model

The ice sheet model used in the calculations is taken from *Weertman* [1976]. Assuming the ice sheet is centered on the pole, the assumption that ice flows as a perfectly plastic solid gives the height of the ice sheet as

$$h(\theta, \theta_0) = \lambda(\theta - \theta_0)^{1/2} \tag{7}$$

The equatorward edge of the ice sheet is at latitude θ_0, where the absorptivity changes. A value of $\lambda = 1000$ m (degree latitude)$^{-1/2}$ gives a central elevation of ~4500 m for an ice sheet extending to 70° latitude.

The annual mean snow budget g can be written $g(\theta) = P(\theta) - A(\theta)$, where P is the annual snowfall and A is the annual ablation. The latitude at which P equals A and g vanishes is called the snow line or equilibrium line, located at θ_e. The parameterization described in section 2 to determine the amount of snow cover assumes that temperature is the most important variable affecting the snow budget. Therefore, the snow line is always assumed to lie at the latitude where the temperature is T_0. *Budyko* [1969] determined from observations that T_0 is -10°C. It can be seen that for snow cover in the absence of an ice sheet $\theta_0 = \theta_e$.

The size of an equilibrium ice sheet, however, is determined by the snow budget of the entire ice sheet, not just the snow budget at one point. This occurs because the flow of ice carries mass from regions of accumulation to regions of ablation. For an ice sheet to be in equilibrium, the snow budget integrated over the ice sheet must vanish. For a zonally averaged model this can be written

$$\int_{\theta_0}^{\pi/2} g(\theta) \cos\theta \, d\theta = 0 \tag{8}$$

This implies that part of the ice sheet must lie in the accumulation zone and part must lie in the ablation zone, equatorward of the equilibrium line. Therefore, the ice sheet edge must be equatorward of the equilibrium line; that is, $\theta_0 < \theta_e$. All other factors being equal, an ice sheet in equilibrium would be larger than a permanent snow field by an amount equal to the ablation zone of the ice sheet.

Two simple methods for parameterizing the size of the ice sheet suggest themselves. One is to place the equilibrium point θ_e at a given temperature; the other is to place the ice margin θ_0 at a given temperature. The latter approach has been adopted for this model. There is no reason to expect that the value of T_0 determined from snow cover observations will be appropriate for an ice sheet. Affixing the ice sheet edge to an isotherm means that the ice sheet size is determined in exactly the same way as snow cover area. This is a useful and simple approximation, and will make the model with the ice sheet identical to that with snow cover, except for the radiative cooling parameterization. Therefore, the edge of the ice sheet lies at the $-10°C$ mean annual isotherm. This has been done to isolate the influence of the infrared cooling parameterization on the sensitivity of the model.

Oerlemans [1980] proposed a model in which the equilibrium point θ_e is affixed to an isotherm. Assuming a form for the mass balance of the ice sheet he finds the latitude of the ice sheet edge θ_0 is proportional to 4/3 θ_e. The factor of 4/3 amplifies the albedo feedback, increasing the sensitivity of the model. This may explain why using the surface temperature in his radiative cooling parameterization did little to change the sensitivity of his model.

In the model described in this and the previous section the meridional flux of heat is computed from the meridional gradient of sea-level temperature. With the assumption of a constant lapse rate this is equivalent to computing the meridional heat transport from the gradient of the mid-tropospheric temperature. The mid-tropospheric temperature gradient may be a more reasonable parameter to think of as controlling the heat transport through baroclinic instability processes.

In the seasonal model of *Pollard et al.* [1980] the meridional heat flux is proportional to the meridional gradient of the sea-level temperature, but the radiative cooling is also only a function of sea-level temperature, despite the inclusion of the ice sheet in the model. The neglect of the height effect and the choice of a value of $b = 1.9$ W m^{-2} °C^{-1} probably lead to larger sensitivity than is warranted. As will be shown in the next two sections, the height effect makes the ice line significantly less sensitive to changes in the solar constant.

4. Analysis of the Model

The effect of the height-dependent terms in (5) on the sensitivity of the energy balance model can be seen by considering the solar heating and radiative cooling terms from (1) together. To combine these two terms, the absorptivity α and the radiative cooling E are written in the following ways:

$$\alpha[T(\theta)] = \alpha_1 - \alpha' H(\theta, \theta_0) \tag{9}$$

and

$$E(T, h) = a_0 + [b_0 + b_1 h H(\theta, \theta_0)]T - \left(b_0 + b_1 h - \frac{a_1}{\Gamma}\right)\Gamma h H(\theta, \theta_0) \tag{10}$$

where $H(\theta, \theta_0)$ is a step function at θ_0. The surface elevation h appears only in the radiative cooling (10), not in the albedo (9). Upon substituting (9) and (10) into (1) and combining terms containing $H(\theta, \theta_0)$, the energy balance equation becomes

$$\frac{D}{\cos\theta}\frac{d}{d\theta}\cos\theta\frac{dT}{d\theta} = -Qs(\theta)\left[\alpha_1 - \left(\alpha' - \frac{(b_0 + b_1 h - (a_1/\Gamma))\Gamma h}{Qs(\theta)}\right)H(\theta, \theta_0)\right] + [a_0 + (b_0 + b_1 h H(\theta, \theta_0))T] \tag{11}$$

This equation can be compared to (2) in *North* [1975]. In (11) the effect of h on the radiative cooling is now split into two parts. The first part has been combined with the albedo in the solar heating term. The second part is the height dependence of b in the radiative cooling term. The first part of the height effect reduces the outgoing radiation calculated in (10) as the surface elevation increases, because $b_0 + b_1 h - a_1/\Gamma$ is positive. In terms of the overall radiation budget of the energy balance model this is equivalent to increasing the solar heating of the ice sheet by reducing the difference in absorptivity α'. We can define the effective absorptivity difference α_e' to be

$$\alpha_e' = \alpha' - \frac{(b_0 + b_1 h - (a_1/\Gamma))\Gamma h}{Qs(\theta)}$$

For the case with clouds at constant pressure levels the following parameter values can be used to estimate α'_e: $\Gamma = 6.5$°C km^{-1}, $h = 3$ km, $Q = Q_0$ and $s \sim 0.16$ at 80° latitude; a_1, b_0 and b_1 were given in section 2. Using these values $\alpha' = .3$, but $\alpha'_e = 0.17$. For the case with clouds at constant σ levels $\alpha'_e = 0.14$. In both cases the height of the ice sheet effectively increases the solar heating of the ice sheet by raising the absorptivity.

The albedo feedback of the model with the ice sheet can now be physically understood in the following way. If the solar constant is reduced slightly, this causes a small cooling and an increase in the size of the ice sheet. The larger ice sheet absorbs less sunlight, but the higher (and colder) surface radiates less, partially counteracting the albedo feedback. The result is some further cooling, but not as much as for snow albedo feedback, where the height effect is absent. The second part of the height effect causes b to increase as the surface elevation increases because b_1 is positive. A larger value of b will also make the model less sensitive. The change in sensitivity of the model caused by including these effects will be shown in the next section by numerical solutions to the energy balance equation.

The analytic solution to the energy balance equation with snow albedo feedback presented by *North* [1975] is valid if the absorptivity is a smooth function of latitude separately on and off the snow-covered area. If b_1 can be neglected, then all the height dependent terms can be included in the effective absorptivity difference. In that case the effective absorptivity of the ice sheet is a smooth function of latitude and the analytic results of North apply.

5. Effects of the Ice Sheet on the Sensitivity of the Model

To illustrate the effect of the ice sheet on sensitivity, the latitude of the ice line θ_0 is computed as a function of solar constant for several cases both with and without the ice sheet. To solve the equation approximately, the diffusion operator, boundary conditions, and forcing are written in finite difference form. Using second-order centered differences for the diffusion operator, (11) becomes the inhomogeneous matrix equation

$$\mathbf{DT} = \mathbf{f}(Q, \theta_0, \cdots) \tag{12}$$

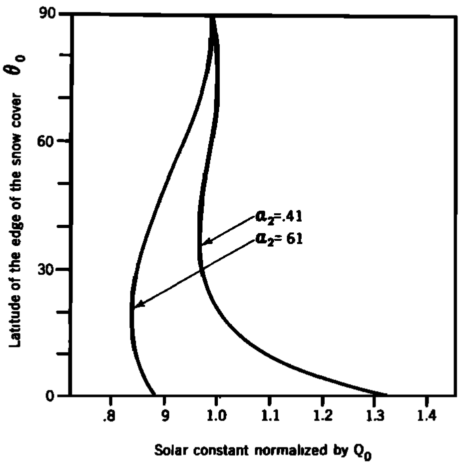

Fig. 6. Equilibrium size of the snow field for two different snow albedos. Raising the snow absorptivity increases the planetary absorptivity and the global mean temperature. This makes the snow cover smaller and also reduces the albedo feedback and the sensitivity of the model.

where **D** is a tridiagonal matrix. Fixing Q, one must find the value of θ_0 such that $T(\theta_0) = T_0$. In practice it is easier to fix θ_0 and iterate to find the Q consistent with the ice line temperature. This method finds both stable and unstable equilibria.

The following values were used for the parameters. The latitudinal distribution of annual mean insolation s was computed assuming a circular orbit and an obliquity of 23.5°. The function s is independent of the season of perihelion and only weakly dependent on the eccentricity. The standard absorptivities were given in section 2. The coefficients used to compute the radiative cooling were taken from the case with clouds at constant pressure levels. Using these parameters the diffusivity D was adjusted to 0.506 W m^{-2} °C^{-1} to give a snow line temperature $T_0 = -10$°C at 70° latitude when the solar constant was Q_0. With the snow line at 70° latitude, the model produces realistic zonal mean temperatures.

In Figure 6 the snowline (no ice sheet) is plotted as a function of solar constant for two different snow absorptivities. In regions where $\partial\theta_0/\partial Q < 0$ the snow line is unstable to small perturbations. In regions where $\partial\theta_0/\partial Q > 0$, the snow line is stable and the sensitivity is proportional to the slope of the equilibrium line. It can be seen in Figure 6 that increasing the absorption by the snow cover (decreasing the albedo) stabilizes the snow line everywhere. From the curve with the 'standard absorption' $\alpha_2 = 0.41$, the reduction in solar constant required to cause an ice covered Earth is ~3%. This is considerably more sensitive than many recent energy balance models that use larger values of b [cf. *Oerlemans and van den Dool,* 1978].

The height of the ice sheet has a significant effect on the sensitivity of the model. This can be clearly seen in Figure 7, which compares the equilibrium curves for the model with and without the ice sheet. The equilibrium curves are of questionable validity equatorward of 40° latitude because the ice sheet becomes unrealistically large and high. The stabilizing effect of the ice sheet can be seen to be very similar to the stabilization induced by increasing the absorptivity of the snow cover in Figure 6. This is in keeping with the analysis in section 4. The model has no stable solutions for ice sheets for the present value of the solar constant, and the solar constant would have to be reduced by ~14% to produce an ice covered earth.

The differences between the two curves could lead to interesting behavior of the ice sheet edge. An example is illustrated in Figure 7. If the solar constant is instantaneously reduced from its present value Q_0 by an amount ΔQ, the snow cover will advance from its present latitude (point 1) to near 50° latitude (point 2). As the snow accumulates, the increase in the surface elevation affects the radiative cooling. The ice sheet size must approach the equilibrium ice sheet size for the reduced value of the solar constant, so the ice sheet shrinks in length as it thickens. The eventual equilibrium icesheet edge lies at about 67° latitude (point 3). If the solar constant is then increased back to Q_0 the ice sheet must vanish, because there are no stable equilibrium ice sheets for $Q = Q_0$. In a time-dependent model with an ice sheet the albedo would vary on time scales determined by the growth rate of snow cover and the shrinkage rate of the ice sheet.

The relative importance of the height dependence of the surface temperature and the height dependence of the radiation coefficients can be seen in the equilibrium curves plotted in Figure 8. Curve A is for snow cover, curve B is for an ice sheet with $a_1 = 2.58$ W m^{-2} km^{-1} and $b_1 = 0.09$ W m^{-2} °C^{-1} km^{-1} and curve C is for an ice sheet with $a_1 = 0$ and $b_1 = 0$. Terms in (11) that include a_1 tend to make the model more sensitive by increasing α'_e. Terms that include b_1 tend to make the model less sensitive both by decreasing α'_e and by increasing b. With a_1 and $b_1 = 0$ the model is slightly more sensitive than with a_1 and $b_1 \neq 0$. Equation (6) allows one to estimate the relative importance of terms involving a_1 and b_1. For temperatures and heights relevant to an ice sheet the terms involving b_1 are larger than the term involving a_1, except for small ice caps. Thus the model is slightly less sensitive when the height dependence of the radiative cooling coefficients is included. The ice sheet model is still considerably less sensitive than the snow model, even when a_1 and $b_1 = 0$. This implies that most of the reduction in the sensitivity is due to the use of surface temperature and is present even if the radiative cooling

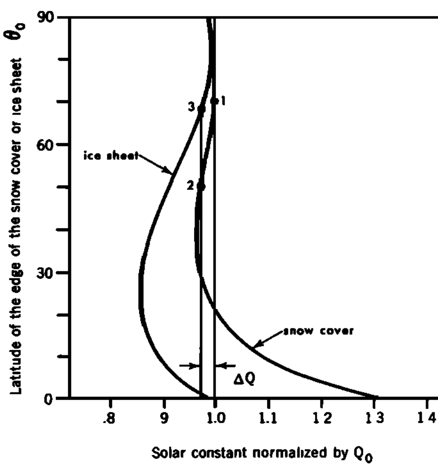

Fig. 7. Equilibrium ice sheet size and equilibrium snow cover size. At any given solar constant the ice sheet is smaller than the snow cover and less sensitive to changes in the solar constant. This is similar to the differences seen in Figure 6. The numbers indicate the possible evolution of the ice sheet size following a sudden decrease in the solar constant.

coefficients do not depend on height. The relative magnitude and even the sign of a_1 and b_1 can change depending on the assumptions made about clouds, as illustrated in section 2. In either case the effect of the surface temperature dominates, and the model with the ice sheet is less sensitive than that with snow cover.

6. CONCLUSIONS

The interactions between climate and large ice sheets are not well understood. Diffusive energy balance models are a good tool for exploring the processes and feedbacks operating in the climate system. In the model proposed here, one new feedback from the ice sheet to the atmosphere is included: the effect of the height of the ice sheet on radiative cooling by the atmosphere. Another significant factor may be the mass budget of the ice sheet. Important feedbacks may exist between the size of the ice sheet, the atmospheric circulation, and the precipitation and ablation at the ice sheet surface. These processes have not been considered here. Instead, the ice sheet is assumed to behave exactly like snow cover. If temperature is the most important factor controlling the size of the ice sheet, the parameterization used here should be useful. A more realistic parameterization of the ice sheet size will require consideration of the mass budget of the ice sheet. Including a mass budget for the ice sheet will require a climate model that has a snow budget.

The effect of the height of the ice sheet on the radiative cooling by the atmosphere has been calculated by using a radiative transfer model. This approach allows greater flexibility than by using satellite observations of the upward IR flux, but significantly different results are obtained depending upon the assumptions made about cloud distributions and lapse rate. The height effects can be put into a diffusive energy balance model using a simple linear parameterization for the radiative cooling as a function of temperature with height dependent coefficients.

The sensitivity of the snow line in an energy balance model has been described before [*Held and Suarez*, 1974; and *North*, 1975]. The sensitivity depends in part on the relative absorptivities of the snow-covered and snow-free areas. One effect of the height of the ice sheet on the radiative cooling can be shown to be equivalent to increasing the absorptivity of the snow-covered area (the ice sheet). This will reduce the effective difference in absorptivity between the snow-covered and snow-free areas. Reducing the absorptivity difference or including the height effect will reduce the sensitivity of the snow line to changes in the solar constant. The other effect of the height is to increase the value of b, the radiative cooling parameter. This will also lead to a more stable model. The height effect can reduce the sensitivity by a factor of 2 or more and should not be neglected when studying the sensitivity of climates with large ice sheets.

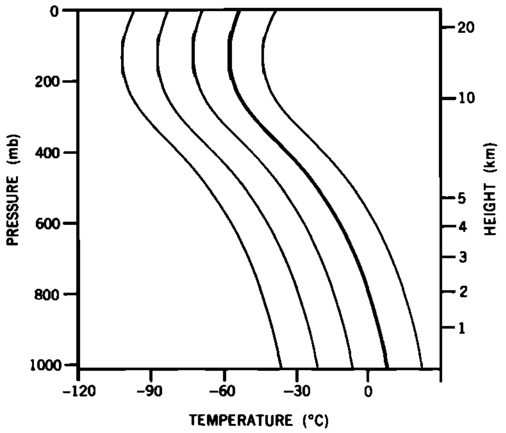

Fig. 9. Climatological temperature from 45°N (heavy line) and perturbed profiles (light lines).

APPENDIX

The form of the temperature perturbation may have a large effect on the relationship between outgoing IR irradiance at the top of the atmosphere and the surface temperature. As an example, if the temperature perturbation is constant with height, the perturbed temperature profiles will be those shown in Figure 9. Calculations were done by using this perturbation for both cloud parameterizations discussed in section 2. The resulting regression coefficients are listed in Table 2. Using this perturbation, cloud top temperatures will increase the same amount as the surface temperature. The radiation emitted by clouds will therefore increase more for a

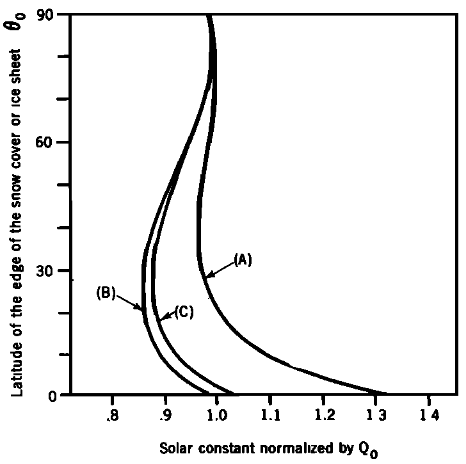

Fig. 8. Equilibrium ice sheet size and equilibrium snow cover size. Curve A is for snow cover. Curve B is for an ice sheet with $a_1 = 2.58$ W m^{-2} km^{-1} and $b_1 = 0.09$ W m^{-2} °C^{-1} km^{-1}, and curve C is for an ice sheet with a_1 and $b_1 = 0$.

TABLE 2. Regression Coefficients for a Linear Fit to the Radiative Cooling E as a Function of Surface Temperature T_*

Surface Elevation h, km	a, W m^{-2}	b, W m^{-2} °C^{-1}
	Clouds at Constant σ Levels	
0	208.2	2.63
1	212.0	2.65
2	215.9	2.66
3	220.1	2.65
4	224.7	2.63
5	230.5	2.71
	Clouds at Constant Pressure Levels	
0	208.2	2.63
1	215.5	2.66
2	226.0	2.67
3	232.6	2.69
4	237.5	2.66
5	244.1	2.75

The climatological temperature profile is perturbed as shown in Figure 9.

given change in the surface temperature than when using the perturbation shown in Figure 4. The coefficient b is therefore larger in every case than those listed in Table 1. In both cloud treatments a increases with height because cloud tops are considerably warmer for the same surface temperature. For the case with clouds at constant pressure levels b does not change as much as with the other perturbation because the total radiation does not depend as much on changes in the surface temperature: temperatures are getting warmer or colder everywhere.

These values of b are even larger than those calculated from satellite observations. A coefficient of this magnitude would tend to make the energy balance model much less sensitive than in the case discussed in section 2.

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